Red-Black Trees

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Outline

- Red-black trees

Summary on Binary Search Trees

- Binary search trees
  - embody the ***divide-and-conquer*** search strategy
  - **Search**, **Insert**, **Min**, and **Max** are $O(h)$, where h is the ***height of the tree***
  - in general, $h(n) = \Omega(\log n)$ and $h(n) = O(n)$
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- Problem
  - worst-case scenario is unlikely but still possible
  - simply bad cases are even more probable

Red-Black Tree

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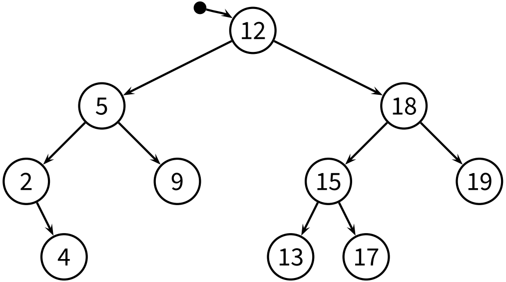

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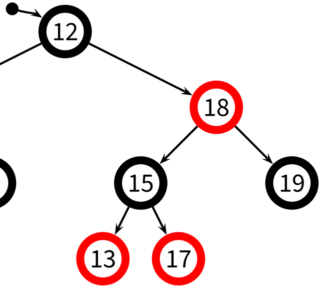

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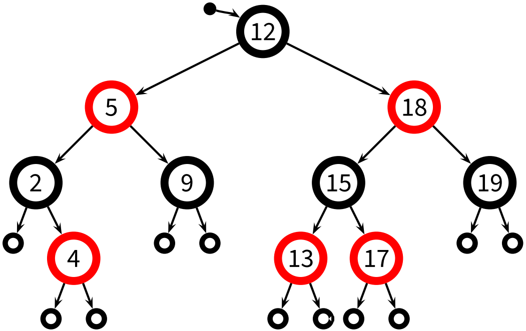

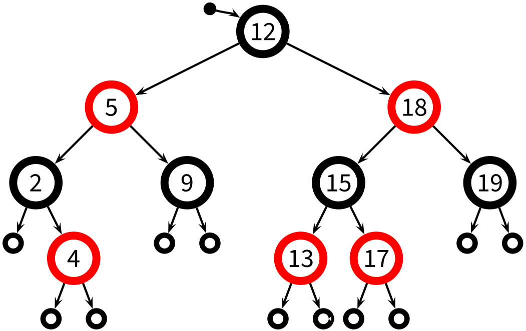

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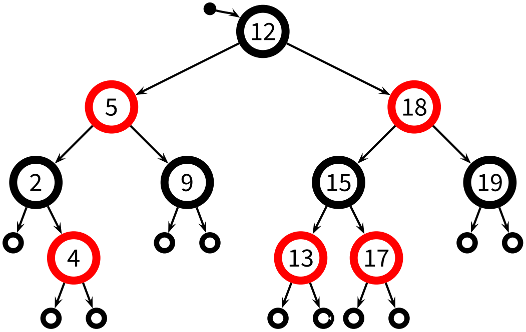

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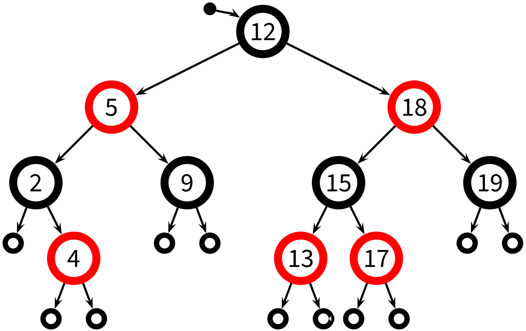

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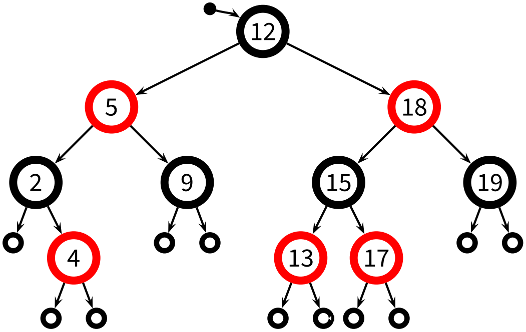

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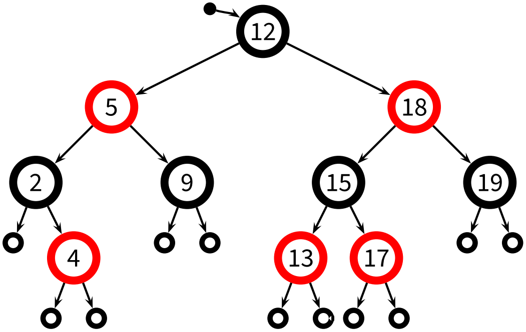

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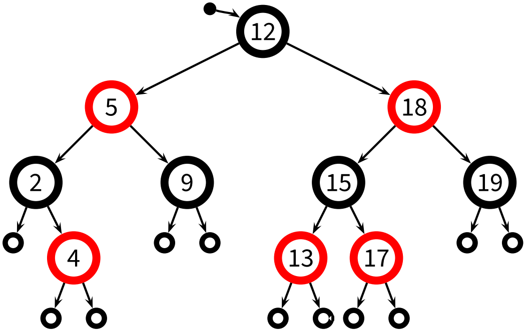

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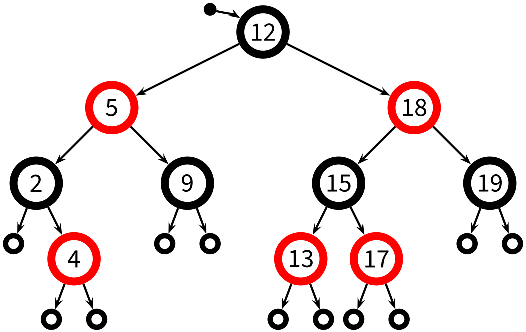

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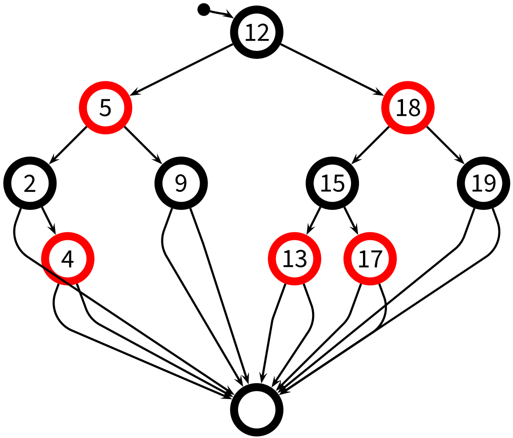

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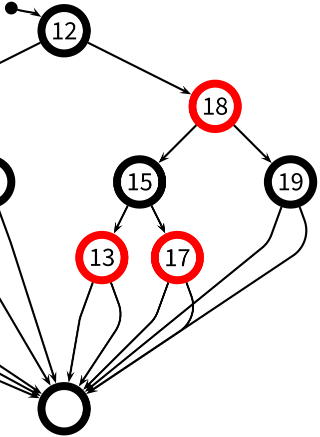

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- the sentinel is also the parent of the root node

Red-Black Tree (3)

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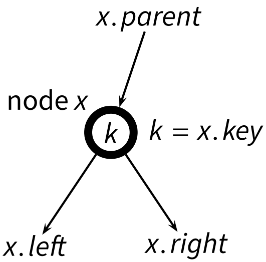

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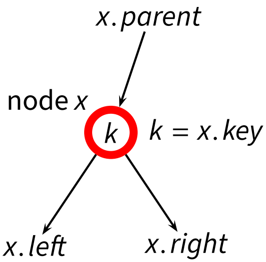

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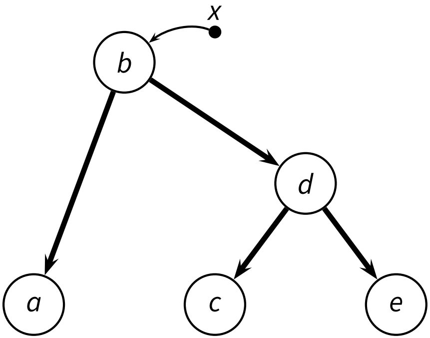
- A red-black tree works as a binary search tree for search, etc.

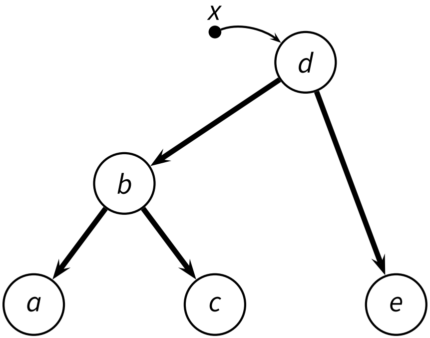
- So, the complexity of those operations is $T(n) = O(h)$, that is

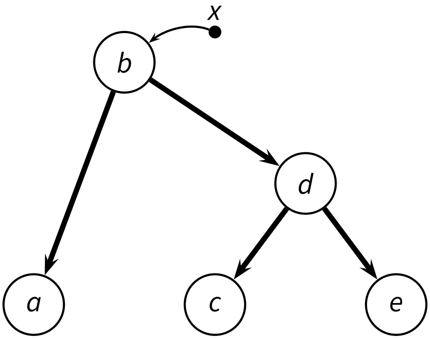
$$\boxed{T(n) = O(\log n)}$$

  - which is also the *worst-case* complexity


x
b
d
a
c
e

Rotation





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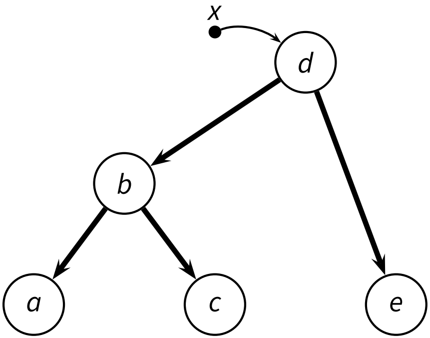

- $x = \textsc{Right-Rotate}(x)$
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- *General strategy*

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- *General strategy*
  1. insert z as in a binary search tree
  2. color z **red** so as to preserve property 5
  3. *fix the tree* to correct possible violations of property 4

RB-Insert

```
RB-INSERT(T, z)
y = T.nil
x = T.root
while x ≠ T.nil
    y = x
    if z.key < x.key
        x = x.left
    else x = x.right
z.parent = y
if y == T.nil
    T.root = z
else if z.key < y.key
    y.left = z
    else y.right = z
z.left = z.right = T.nil
z.color = RED
RB-INSERT-FIXUP(T, z)
```

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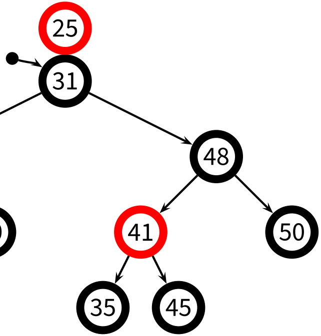

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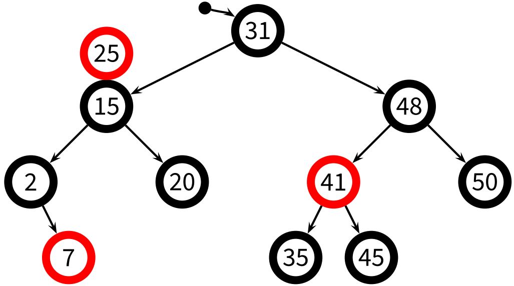

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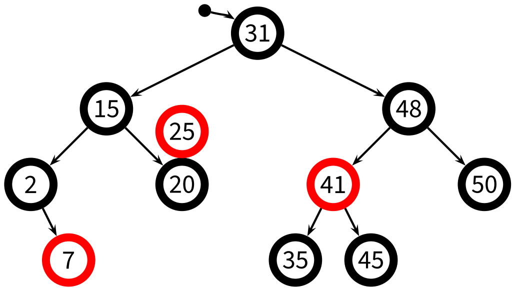

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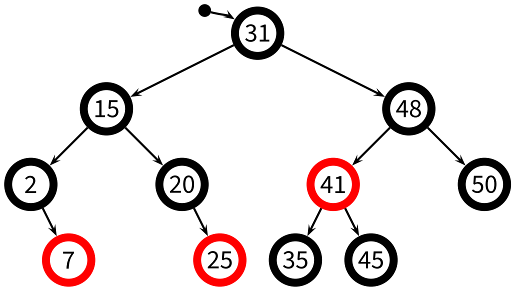

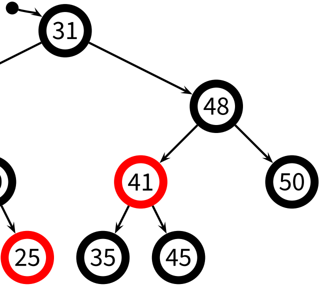

- *z*'s parent is **black**, so no fixup needed

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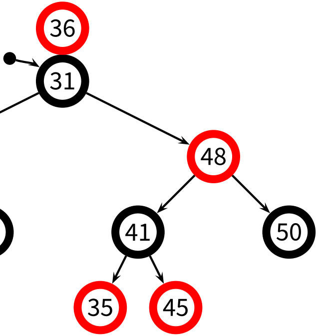

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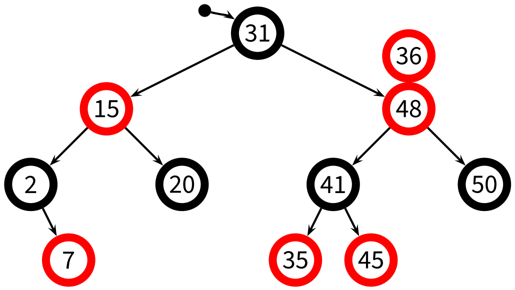

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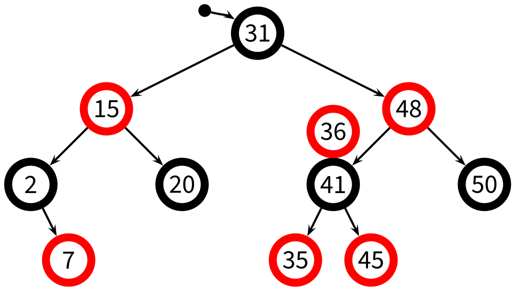
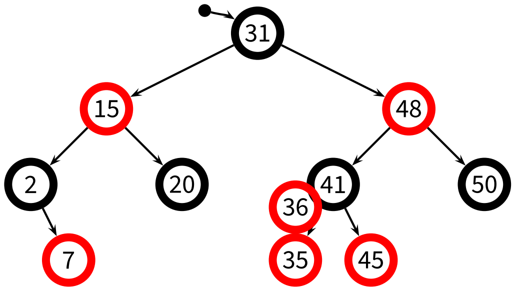
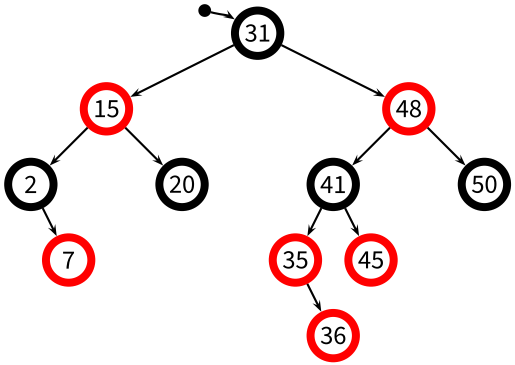


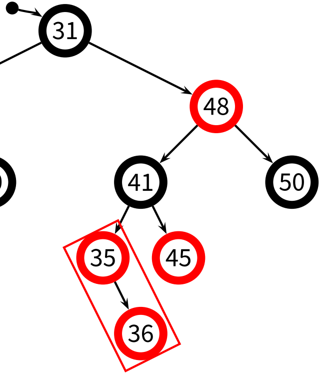
Red-Black Insertion (3)



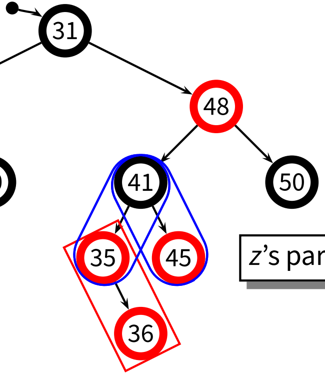
Red-Black Insertion (3)



Red-Black Insertion (3)

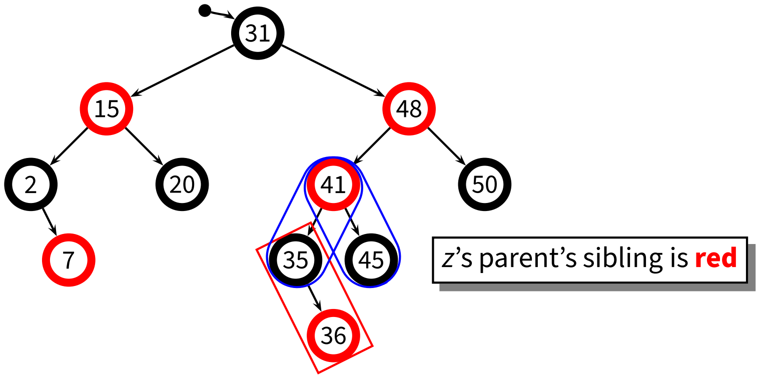


Red-Black Insertion (3)

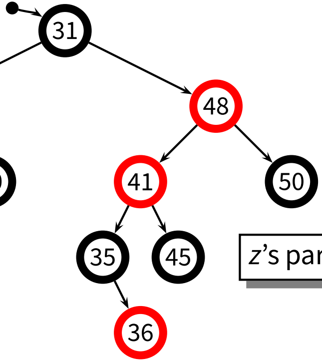


z's parent's sibling is **red**

Red-Black Insertion (3)

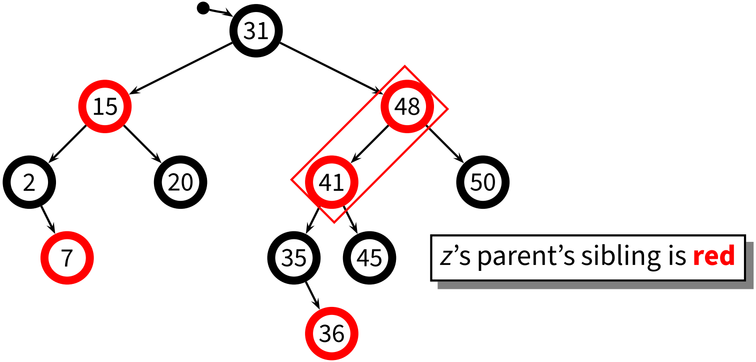


Red-Black Insertion (3)

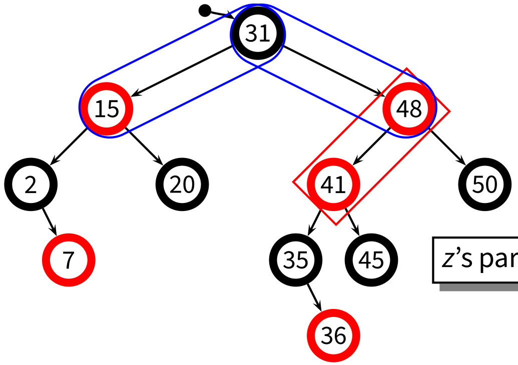


z's parent's sibling is **red**

Red-Black Insertion (3)

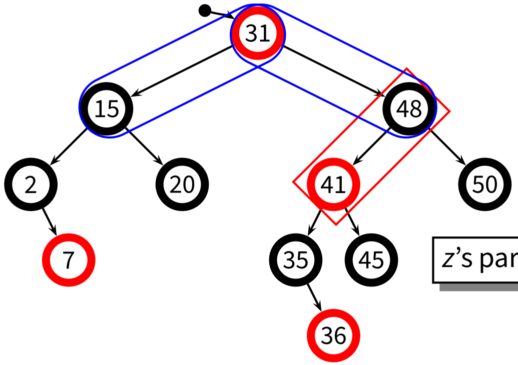


Red-Black Insertion (3)



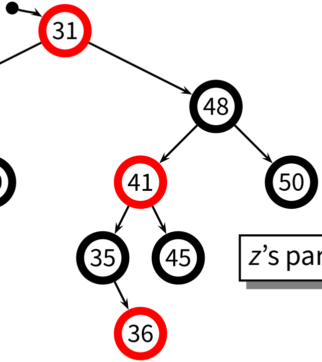
z's parent's sibling is **red**

Red-Black Insertion (3)



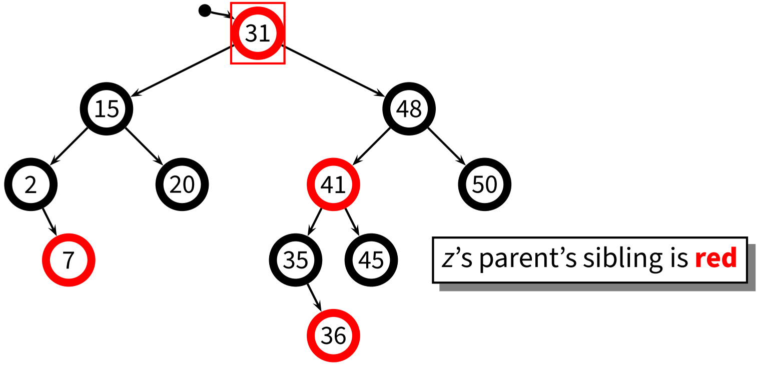
z's parent's sibling is **red**

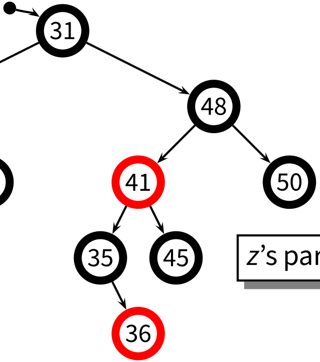
Red-Black Insertion (3)



z's parent's sibling is **red**

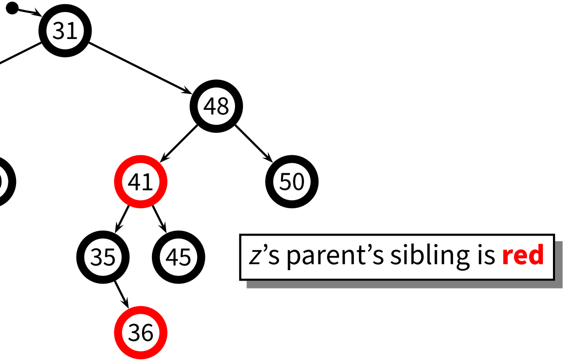
Red-Black Insertion (3)





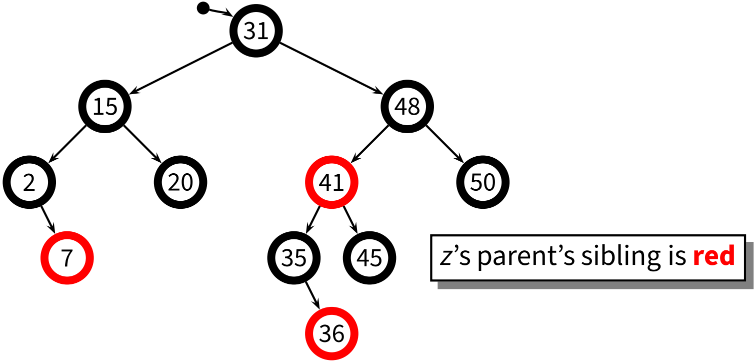
z's parent's sibling is **red**

Red-Black Insertion (3)



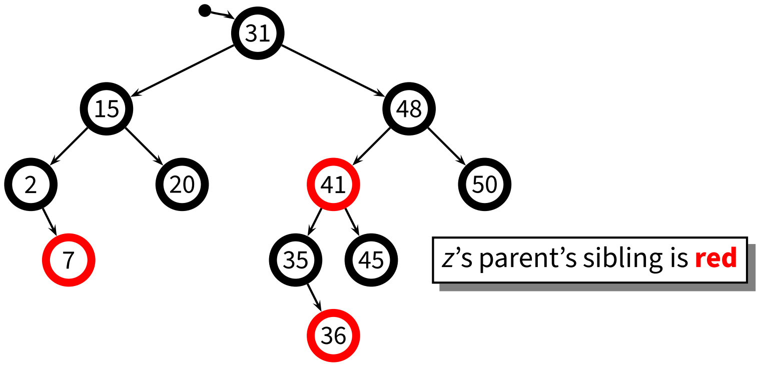
- A **black** node can become **red** and transfer its **black** color to its two children

Red-Black Insertion (3)



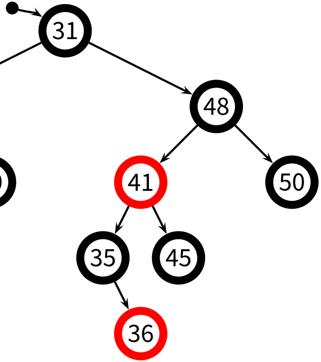
- A **black** node can become **red** and transfer its **black** color to its two children
- This may cause other **red**–**red** conflicts, so we iterate…

Red-Black Insertion (3)

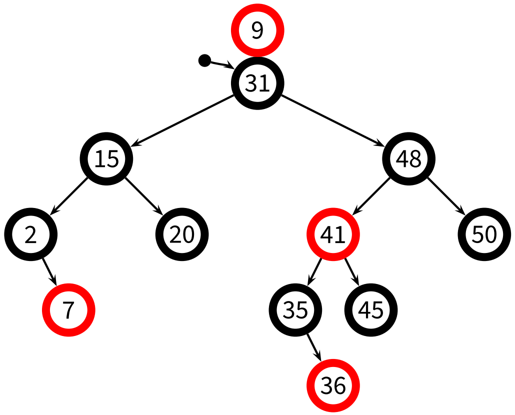


- A **black** node can become **red** and transfer its **black** color to its two children
- This may cause other **red**–**red** conflicts, so we iterate…
- The root can change to **black** without causing conflicts

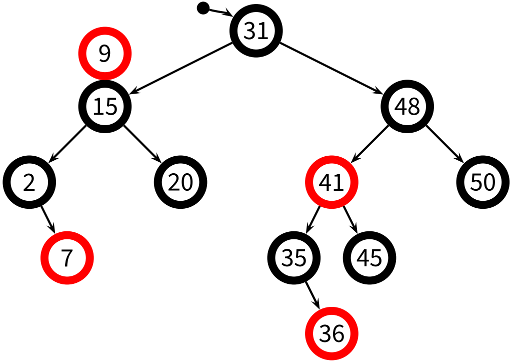
Red-Black Insertion (4)



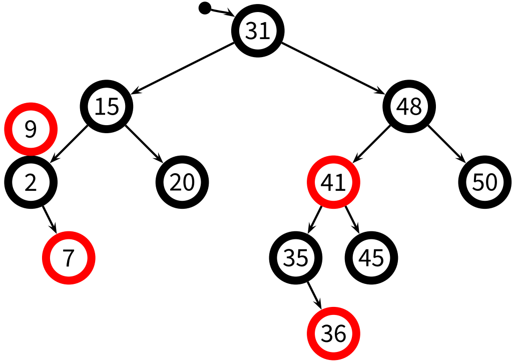
Red-Black Insertion (4)



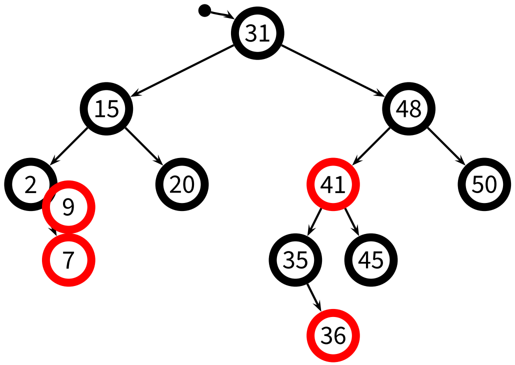
Red-Black Insertion (4)



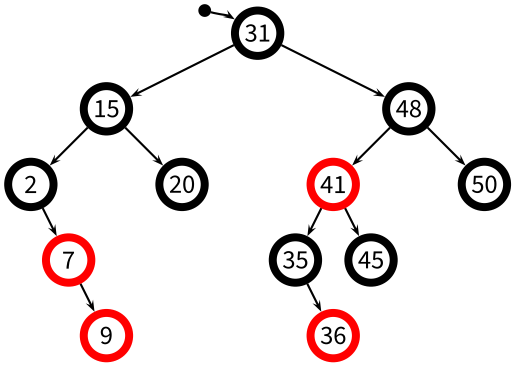
Red-Black Insertion (4)



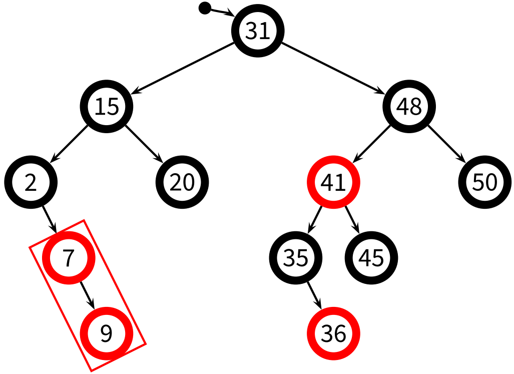
Red-Black Insertion (4)



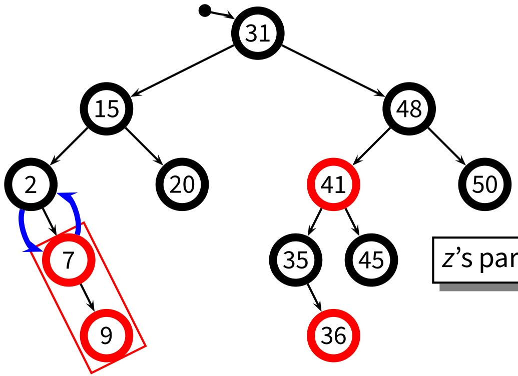
Red-Black Insertion (4)



Red-Black Insertion (4)

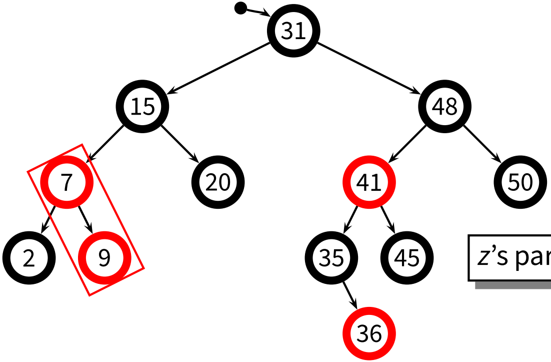


Red-Black Insertion (4)



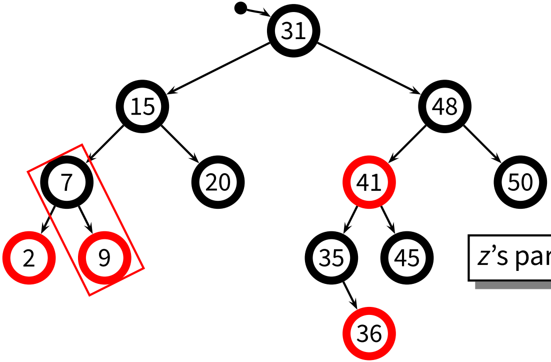
z's parent's sibling is **black**

Red-Black Insertion (4)



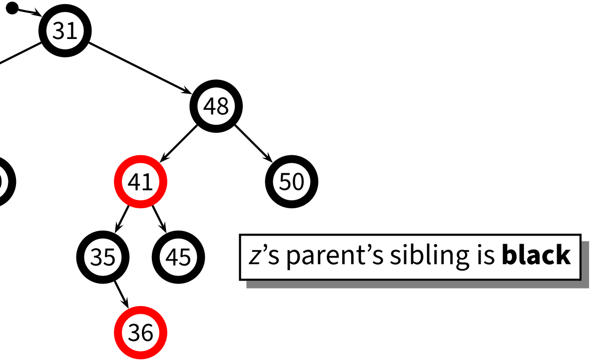
z’s parent’s sibling is **black**

Red-Black Insertion (4)



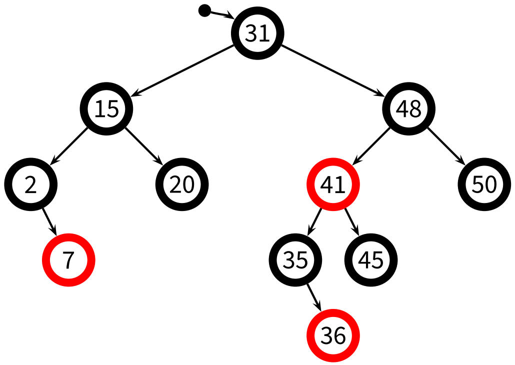
z's parent's sibling is **black**

Red-Black Insertion (4)

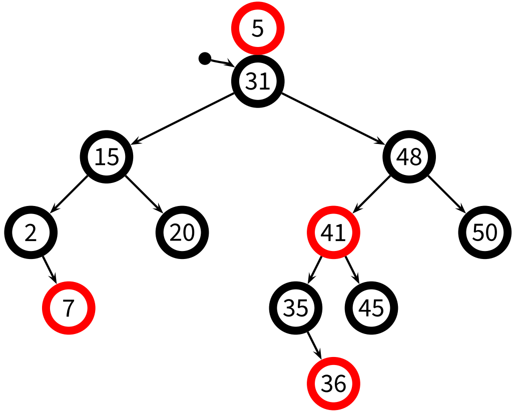


- An *in-line* ***red***–**red** conflicts can be resolved with a rotation plus a color switch

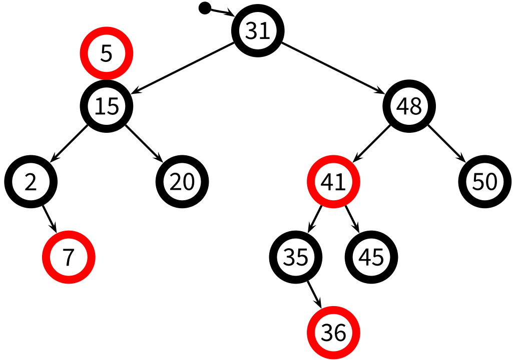
Red-Black Insertion (5)



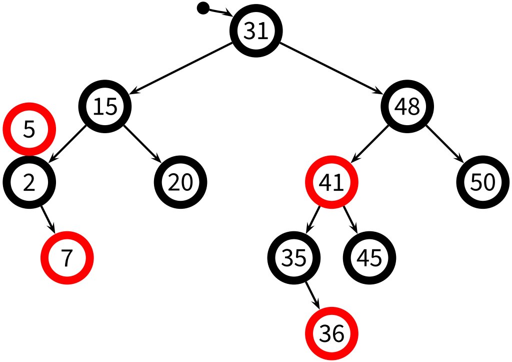
Red-Black Insertion (5)



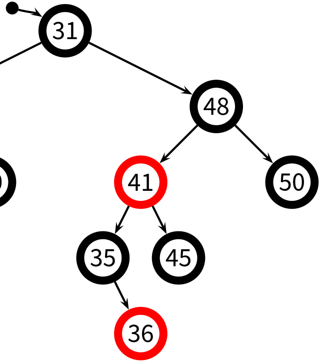
Red-Black Insertion (5)



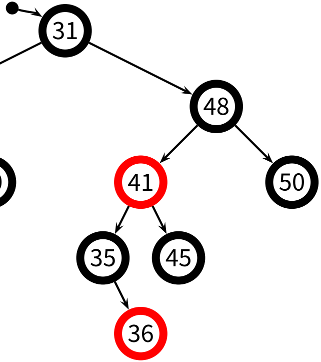
Red-Black Insertion (5)



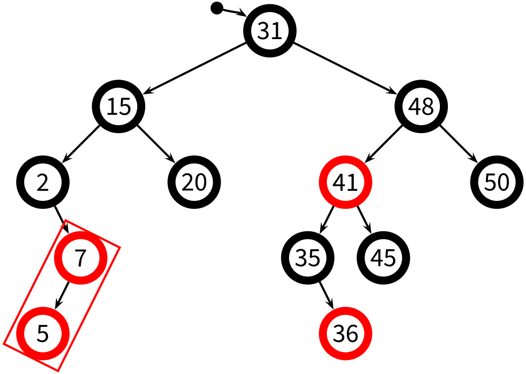
Red-Black Insertion (5)



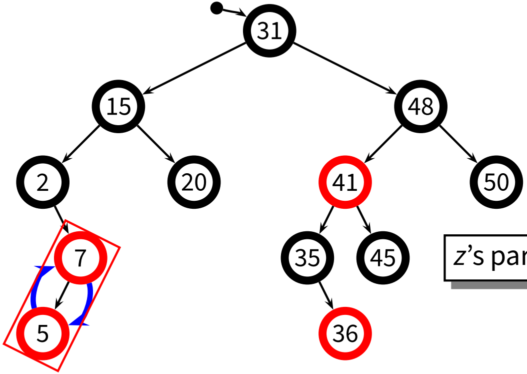
Red-Black Insertion (5)



Red-Black Insertion (5)

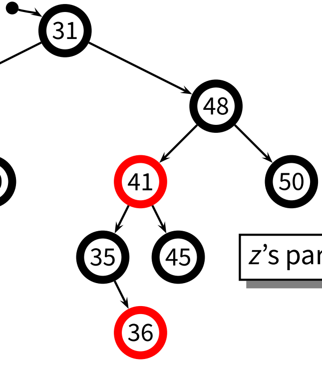


Red-Black Insertion (5)



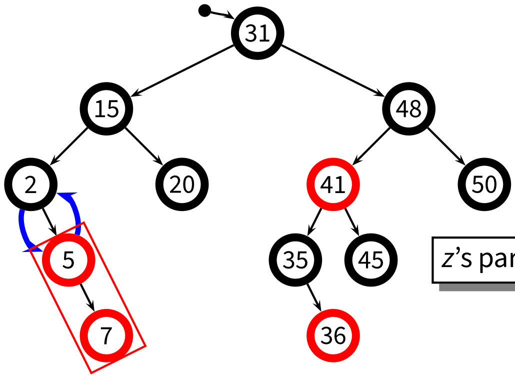
z’s parent’s sibling is **black**

Red-Black Insertion (5)



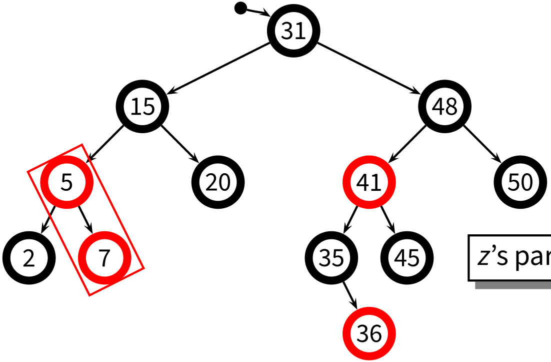
z's parent's sibling is **black**

Red-Black Insertion (5)

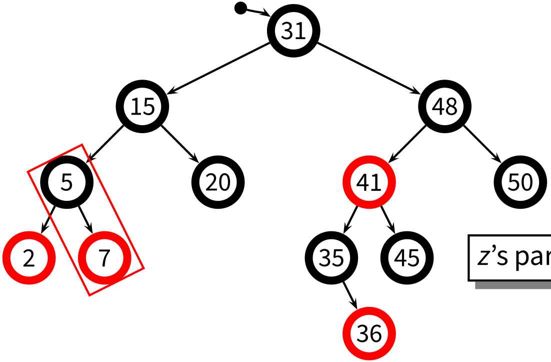


z's parent's sibling is **black**

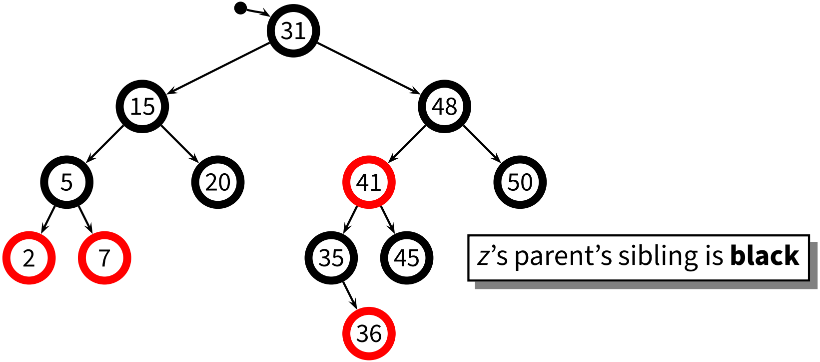
Red-Black Insertion (5)



z's parent's sibling is **black**



z's parent's sibling is **black**



- A *zig-zag* **red–red** conflicts can be resolved with a rotation to turn it into an *in-line* conflict, and then a rotation plus a color switch